T-equivariant cohomology of cell complexes and the case of infinite Grassmannians

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Abstract. In 1998, Goresky, Kottwitz, and MacPherson showed that for certain spaces X equipped with a torus action, the equivariant cohomology ring $H^*_T(X)$ can be described by combinatorial data obtained from its orbit decomposition. Thus, their theory transforms calculations of the equivariant topology of X to those of the combinatorics of the orbit decomposition. Since then, many authors have studied this interplay between topology and combinatorics. In this paper, we generalize the theorem of Goresky, Kottwitz, and MacPherson to the (possibly infinite-dimensional) setting where X is any equivariant cell complex with only even-dimensional cells and isolated T-fixed points, along with some additional technical hypothesess on the gluing maps. This generalization includes many new examples which have not yet been studied by GKM theory, including homogeneous spaces of a loop group LG.

1 Introduction and Background

The main purpose of this paper is to describe the equivariant cohomology of homogenous spaces of some affine Kac-Moody groups. Among these examples are the spaces of based loops, ΩK, considered as a coadjoint orbit of the extended loop group $\widehat{LK} \rtimes S^1$. The space ΩK is a symplectic Banach manifold, and the maximal torus $T \subseteq \widehat{LK} \rtimes S^1$ acts on ΩK in a Hamiltonian fashion. This Hamiltonian system exhibits many properties familiar in symplectic geometry: its moment image is convex [1, 14], and its T-fixed points are isolated. Hence our motivation is to extend, to these infinite-dimensional examples, results in finite-dimensional symplectic geometry that compute equivariant cohomology. Although the examples that motivate us come from symplectic geometry, our proofs rely heavily on techniques from algebraic topology.

We now describe the specific symplectic-geometric results that we will generalize. Let X be a compact equivariantly formal[4] T-space, where $T = (S^1)^n$ is a finite-dimensional torus. A theorem of Goresky, Kottwitz, and MacPherson, which we call "the GKM theorem" in honor of its authors,

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MSC 2000 Subject Classification: Primary: 55N91 Secondary: 22E65, 53D20
Keywords: equivariant cohomology, cell complexes, graphs, affine Kac-Moody groups

[4]Equivariant formality is a technical assumption. It comes for free in all the examples we consider.

gives a combinatorial description of the equivariant cohomology ring $H_T^*(X;\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 [4]. The field coefficients here are crucial.

We define the k-stratum[5] $X^{(k)}$ of X to be

$$X^{(k)} := \{x \in X \mid \dim(T \cdot x) \leq k\}.$$

Thus, the 0-stratum $X^{(0)}$ is just the set of fixed points X^T. This gives the *orbit decomposition* of X. In the GKM theorem, we pay particular attention to the 0-stratum and the 1-stratum, on which additional hypotheses are made. Note that $X^{(l)} \subseteq X^{(k)}$ for $l \leq k$, so the fixed points X^T are contained in the 1-stratum. In the situation that Goresky, Kottwitz and MacPherson consider, the fixed points are isolated, the equivariant cohomology $H_T^*(X)$ is a free $H_T^*(pt)$-module, and the kernel of the restriction map $H_T^*(X) \to H_T^*(X^T)$ is a torsion submodule. Therefore, this restriction is an injection into the T-equivariant cohomology of the fixed point set $H_T^*(X^T)$. It is important to note that the equivariant cohomology of X^T, a finite set of isolated points, is simply the direct product of polynomial rings:

$$H_T^*(X^T;\mathbb{F}) = \prod_{p \in X^T} H_T^*(pt;\mathbb{F}) \cong \prod_{p \in X^T} \mathbb{F}[x_1, \ldots, x_n],$$

where the degree of x_i is 2. The x_i are naturally identified with characters of T and $H_T^*(pt)$ with the symmetric algebra on the weight lattice Λ.

The GKM theorem [4] now asserts that the image of $H_T^*(X)$ in $H_T^*(X^T)$ can be described by simple combinatorial data involving the orbit decomposition of X. The hypotheses on X ensure that the 1-stratum consists only of 2-spheres. These spheres are rotated by T with a weight $\alpha \in \mathfrak{t}^*$ (defined up to sign), and have two fixed points. They can only intersect at fixed points. Using this data, we associate to this T-space X a graph $\Gamma = (V, E)$, with vertex set $V = X^T$, and edges joining two vertices if they are the two fixed points on one of the 2-spheres. Moreover, we associate to each edge e a weight α_e which is precisely the weight specifying the action of T on the corresponding 2-sphere. Note that we may think of α_e as a linear polynomial (i.e. degree 2 class) in $H_T^*(pt)$. The image of $H_T^*(X)$ depends only on this graph Γ and the isotropy data α_e's. The GKM theorem says that

$$H_T^*(X;\mathbb{F}) \cong \left\{ f : V \to H_T^*(pt;\mathbb{F}) \;\middle|\; \begin{array}{l} f(p) - f(q) = \alpha_e \cdot g \\ \text{for every } e = (p,q) \in E \\ \text{and some } g \in H_T^*(pt;\mathbb{F}) \end{array} \right\}.$$

In other words, to each vertex p we assign a polynomial $f(p)$. These polynomials must satisfy some compatibility conditions according to the edge weights. Namely, if $e = (p, q)$ is an edge with weight α_e, then $f(p) - f(q)$ must be a multiple of α_e. We now give a simple example.

Let $\mathcal{O}_\lambda$ be a generic coadjoint orbit of $SU(3)$. Then the maximal torus T^2 acts on $\mathcal{O}_\lambda$ by conjugation. There are six fixed points, the one-stratum is 2-dimensional, and the associated GKM graph is shown in Figure 1. Other examples of GKM spaces include toric varieties and coadjoint orbits of any semisimple Lie group. An identical description can also be given for the equivariant

[5] In the symplectic geometry literature, the space $X^{(k)}$ is usually referred to as the k-skeleton. We will use this term in the context of cell complexes, so we are introducing the word stratum to avoid confusion.

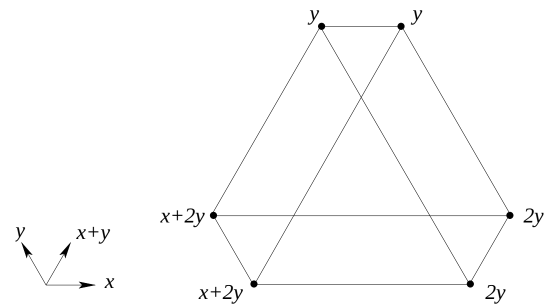

Figure 1: This shows the GKM graph for a generic coadjoint orbit $\mathcal{O}_\lambda$ of $SU(3)$. The weights are indicated in the lower left of the figure. There is a polynomial attached to each vertex, and the polynomials satisfy the compatibility conditions, so this does represent an equivariant cohomology class of $H_T^*(\mathcal{O}_\lambda; \mathbb{Z})$.

cohomology of hypertoric varieties [6]. Computations in the equivariant cohomology ring of flag varieties are closely related to Schubert calculus, and the GKM description of this ring has added new insights to this field (see, for example, [3, 5]).

The remainder of the article is organized as follows. In Sections 2 and 3, we generalize the results of Goresky, Kottwitz and MacPherson to equivariant cell complexes X, possibly infinite-dimensional. We also allow more general coefficient rings R. There are two results which we must prove. The first is the injectivity of $H_T^*(X) \to H_T^*(X^T)$, which holds when X has only even-dimensional cells; we prove this in Section 2. The second result is the generalized GKM theorem, combinatorially describing the image of $H_T^*(X)$ in the cohomology of the fixed points. This theorem is more subtle, and requires additional hypotheses on X. We prove this in Section 3. In Section 4, we give a canonical choice of module generators for $H_T^*(X)$. In Section 5 we discuss the examples of homogeneous spaces for some affine Kac-Moody groups, which include the example of ΩK mentioned above.

Acknowledgments. We are grateful to Allen Knutson for suggesting the problem of looking at the possible GKM theory for the homogeneous spaces of loop groups and for teaching the first and second author how to draw GKM pictures. The first and third authors thank Jonathan Weitsman for many useful discussions. The first author thanks Robert Wendt for straightening out all that is twisted (and untwisted) in affine Lie algebras.

The third author was supported in part by a National Science Foundation Postdoctoral Fellowship. All authors are grateful for the hospitality of the Erwin Schrödinger Institute in Vienna, where some of this work was conducted.

2 The injectivity theorem for cell complexes

We show in this section that the equivariant cohomology of X injects into the equivariant cohomology of its fixed points X^T. Note that in the category of finite-dimensional symplectic manifolds

with Hamiltonian torus action, this result is a familiar theorem of Kirwan [8]. However, in the more general setting of a cell complex with T-action, we need a separate argument. This result is contained in Theorem 2.2.

We begin with a technical lemma that characterizes the kernel of the restriction map $H_T^*(Y;R) \to H_T^*(Y^T;R)$.

Lemma 2.1 *Let Y be a finite dimensional T-space with finitely many orbit types T/G_i. Let R be a ring whose torsion is coprime to the orders of the groups $\pi_0(G_i)$. Then the kernel of the restriction map*

$$H_T^*(Y;R) \longrightarrow H_T^*(Y^T;R)$$

is a torsion $H_T^*(pt;R)$-module.

Proof: Let $Y^T = Y^0 \subset Y^1 \subset Y^2 \subset \cdots \subset Y^m = Y$ be a filtration of Y so that $Y^i \setminus Y^{i-1}$ has a single orbit type T/G_i. We get a spectral sequence from this filtration, with

$$E_1^{p,q} = H_T^{p+q}(Y^p, Y^{p-1};R) \Longrightarrow H_T^{p+q}(Y;R).$$

The edge homomorphism $H_T^n(Y;R) \to E_\infty^{0,n} \hookrightarrow E_1^{0,n} = H_T^n(Y^0;R)$ is the restriction map we are interested in. We will prove the lemma by showing that this map becomes an isomorphism after tensoring with the field of fractions F of $H_T^*(pt;R)$. This is because $E_1^{p,q} \otimes F = 0$ for all $p \geq 1$, and so the spectral sequence collapses after tensoring with F. To show that $E_1^{p,q} = H_T^{p+q}(Y^p, Y^{p-1};R)$ is torsion for all $p \geq 1$, we consider the diagram

$$\begin{tikzcd}
BT \arrow[d, "\pi"'] & (Y^p \times_T ET, Y^{p-1} \times_T ET) \arrow[l] \arrow[d] & \\
B(T/G_p) & (Y^p \times_{T/G_p} E(T/G_p), Y^{p-1} \times_{T/G_p} E(T/G_p)) \arrow[l, "\varrho"'] \arrow[r, "\cong"] & (Y^p/T, Y^{p-1}/T)
\end{tikzcd}$$

Let $x \in H^2(B(T/G_p);R)$ be one of the generators. Its preimage $\pi^*(x)$ is not a zero-divisor in $H^2(BT;R)$ by the coprimality assumption. Since Y^p/T is finite dimensional, we know that $\varrho^*(x)$ is nilpotent. Therefore, $\pi^*(x)$ acts nilpotently on

$$H^*(Y^p \times_T ET, Y^{p-1} \times_T ET;R) = E_1^{p,q},$$

completing the proof. □

We now turn our attention to cell complexes. We say that a space X has a T-invariant cell decomposition if X can be built by successively attaching cells via T-equivariant maps. Each cell has only finitely many orbit types. We do not require the attaching map of a cell to map the boundary to smaller dimensional cells. We now state the injectivity result.

Theorem 2.2 *Let X be a space with an action of a finite-dimensional torus T, and a T-invariant cell decomposition with only even-dimensional cells, finitely many in each dimension. For any stabilizer group G of a point, suppose that R is a ring whose torsion is coprime to the order of the*

group $\pi_0(G)$. Let $\iota : X^T \hookrightarrow X$ denote the inclusion map. Then the pullback

$$\iota^* : H_T^*(X; R) \to H_T^*(X^T; R)$$

is an inclusion.

Proof: We proceed by proving a series of claims. The main idea of the proof is to take a non-zero class, restrict it to a *finite* T-equivariant cell complex, where we will be able to apply Lemma 2.1 to conclude injectivity. Finding the appropriate finite T-equivariant cell complex requires some knowledge of the module structure of $H_T^*(X; R)$. We begin by determining this.

Claim 2.3 *$H_T^*(X; R)$ is a free $H_T^*(pt; R)$-module, with one generator[6] in degree $2k$ for each cell of dimension $2k$ for all $k \geq 0$.*

Proof: We first show that $H_T^*(X; R)$ is a free $H_T^*(pt; R)$-module. Let X_p denote the space built out of the first p cells. Consider the cofibration $X_p \to X_{p+1} \to S^{2n}$. By induction, $H_T^*(X_p)$ is evenly graded. By degree considerations, the long exact sequence of this cofibration splits into short exact sequences

$$0 \to \tilde{H}_T^*(S^{2n}) \to H_T^*(X_{p+1}) \to H_T^*(X_p) \to 0.$$

Again by induction, $H_T^*(X_p)$ is free, therefore $H_T^*(X_{p+1}) \cong \tilde{H}_T^*(S^{2n}) \oplus H_T^*(X_p)$. Now we must prove that $\tilde{H}_T^*(S^{2n})$ is a free module with one generator of degree $2n$.

We will compute $\tilde{H}_T^*(S^{2n}; R) = H^*(S^{2n} \times_T ET, BT; R)$ using the Serre spectral sequence. This has $E_2^{k,\ell} = H^k(BT; \tilde{H}^\ell(S^{2n}; R))$, which is non-zero only when $\ell = 2n$. Thus, this sequence collapses at the E_2 term, since there is only one non-zero row, and therefore $\tilde{H}_T^*(S^{2n}; R)$ is $H_T^{*-2n}(pt; R)$.

Now we use the Milnor sequence

$$0 \to \varprojlim{}^1 \, H^{*-1}(X_i) \to H^*(\varinjlim X_i) \to \varprojlim \, H^*(X_i) \to 0,$$

where $\varprojlim^1$ denotes the first (and only non-zero) derived functor of the inverse limit functor. To finish the argument, we must check that the $\varprojlim^1$ is zero. This is true because all the maps $H_T^*(X_{p+1}) \to H_T^*(X_p)$ are surjective, and the inverse limit of surjective maps is exact. We may now conclude that $H_T^*(X; R)$ is a free $H_T^*(pt; R)$-module, with one generator for each cell. $\diamondsuit$

Using the above claim, we may think of each class $\kappa \in H_T^*(X; R)$ as an element of $H_T^*(pt; R)$, attached to each cell. Note, however, that the isomorphism between $H_T^*(X)$ and the sum $\bigoplus H_T^*(pt)$ (one summand for each cell) is not canonical since it relied on choosing splittings of the projections $H_T^*(X_{p+1}) \to H_T^*(X_p)$.

Let $\kappa \in H_T^*(X; R)$ be a non-zero equivariant class. Our goal is to show that $\iota^*(\kappa) \in H_T^*(X^T; R)$ is also non-zero. As an intermediate step, we restrict our attention to an appropriate finite T-equivariant sub-cell complex Y of X.

Claim 2.4 *If $\kappa \in H_T^*(X; R)$ is non-zero, then there exists a finite T-equivariant sub-cell complex Y, with inclusion $r : Y \hookrightarrow X$, such that $r^*(\kappa) \in H_T^*(Y; R)$ is non-zero.*

[6]If there are infinitely many cells in a given dimension, all our Theorems still hold, by replacing the phrase "free $H_T^*(pt)$-module" with "direct product of free rank 1 $H_T^*(pt)$-modules."

Proof: For any T-equivariant sub-cell complex Y of X, the same argument as above shows that $H_T^*(Y;R)$ is a free $H_T^*(pt;R)$-module, generated by its cells. The inclusion $r : Y \hookrightarrow X$ induces a projection $H_T^*(X;R) \to H_T^*(Y;R)$, which can be interpreted as mapping to zero those generators corresponding to cells in X but not in Y, and is the identity on the remaining generators.

If $\kappa \in H_T^*(X;R;R)$ is a non-zero class, then there exists a cell C in X such that the equivariant number of κ on that cell is non-zero. Let Y be a finite sub-cell complex containing that cell. Then $r^*(\kappa)$ is non-zero in $H_T^*(Y;R)$. $\diamondsuit$

Having restricted our attention to a finite T-equivariant cell complex Y, we may now apply Lemma 2.1.

Claim 2.5 *Let Y be a finite T-equivariant cell complex with only even-dimensional cells, and let $\iota_Y : Y^T \hookrightarrow Y$ denote the inclusion map. Then the pullback*

$$\iota_Y^* : H_T^*(Y;R) \to H_T^*(Y^T;R)$$

is an inclusion.

Proof: By Claim 2.3, $H_T^*(Y;R)$ is a free $H_T^*(pt;R)$-module. By Lemma 2.1, the kernel of ι_Y^* is a torsion submodule of $H_T^*(Y;R)$. Therefore, in this case the kernel must be trivial. $\diamondsuit$

We will now show that κ must have non-zero image in $H_T^*(X^T;R)$. Choose Y such that the restriction of κ to $H_T^*(Y;R)$ is non zero. We have the following commutative diagram

$$\begin{tikzcd} H_T^*(X^T;R) \arrow[r] & H_T^*(Y^T;R) \\ H_T^*(X;R) \arrow[r] \arrow[u] & H_T^*(Y;R). \arrow[u] \end{tikzcd}$$

Since κ restricts to be a non-zero class in $H_T^*(Y^T;R)$ it also restricts to be non-zero in $H_T^*(X^T;R)$. Hence $H_T^*(X;R)$ injects into $H_T^*(X^T;R)$. This completes the proof of Theorem 2.2. $\square$

Remark 2.6 In the above argument, an important step was to show the existence of the finite-dimensional sub-cell-complex Y. It is false, in general, to claim for a non-zero cohomology class c that there exists a finite-dimensional sub-cell-complex Y to which c restricts nontrivially. The problem is that the natural map $H^*(\varinjlim X_i) \to \varprojlim H^*(X_i)$ is not injective in general. Instead, one has the Milnor sequence [10]

$$0 \to \varprojlim{}^1 H^{*-1}(X_i) \to H^*(\varinjlim X_i) \to \varprojlim H^*(X_i) \to 0.$$

A simple example where this $\varprojlim^1$ shows up is in the computation of $H^2(K(\mathbb{Z}[\frac{1}{p}],1),\mathbb{Z}) = \mathbb{Z}_p/\mathbb{Z}$, where the Eilenberg-MacLane space $K(\mathbb{Z}[\frac{1}{p}],1)$ is taken to be the direct limit of spaces homotopy equivalent to S^1, and mapping into each other via cofibrations that induce multiplication by p on H^1.

However, in our case, the maps $X_i \to X_{i+1}$ always induce surjections in cohomology, and therefore the $\varprojlim^1$ term always vanishes.

3 The GKM theorem for cell complexes

We now show that the image of the equivariant cohomology of X in $H_T^*(X^T;R)$ can be identified by simple combinatorial restrictions involving the T-action and the gluing maps. This is the content of Theorem 3.4. The injectivity result of the previous section is quite general. We must now make some additional assumptions on X in order to make this GKM computation.

Assumption 1 The space X can be equipped with a T-invariant cell decomposition, with only even dimensional cells and only finitely many in each dimension.

Assumption 2 We identify each cell D^{2n} with the unit disc in $\mathbb{C}^n$. Under this identification, the torus action on D^{2n} is a linear action, given by a group homomorphism $T \to T^n$, where the T^n-action on $\mathbb{C}^n$ is the standard action.

Assumption 3 The weights $\{\alpha_i\}$ of the T action on each cell D^{2n} are pairwise relatively prime as elements of the polynomial ring $H_T^*(pt;R) \cong R[x_1,\ldots,x_k]$, where $k = \dim(T)$. In other words, if $\alpha_i|\gamma$ for all i, then $\prod \alpha_i \Big| \gamma$, where $\gamma \in H_T^*(pt;R)$. Moreover, the α_i are not zero divisors.

Assumption 4 Let W denote the cell complex of the first $i-1$ cells, and let D^{2n} denote the i^{th} cell. Let $\phi : \partial D^{2n} \to W$ be the attaching map for a cell D^{2n}. Then for each $D^2 \subseteq D^{2n}$ corresponding to an eigenspace of the T-action, $\phi(\partial D^2) \subseteq W^T$, i.e. the boundary of each D^2 must be mapped to a fixed point of one of the earlier cells.

In Assumption 1, we could have merely assumed that X was a CW complex, but this would exclude lots of interesting examples coming from symplectic geometry, e.g. toric varieties. Indeed, in those examples, the Morse functions used to define the cell structures are often not Morse-Smale and as a consequence, the cell decompositions are not CW complexes.

In Assumption 3 we use the identification of the weight lattice Λ with the degree 2 elements $H_T^2(pt;R)$. Thus it makes sense to ask that two weights $\alpha, \alpha' \in \Lambda \cong H_T^2(pt;R)$ be relatively prime in the ring $H_T^*(pt;R)$. Assumptions 2 and 3 imply that the T fixed points of X are isolated, and there is exactly one fixed point for each cell in the cell decomposition. By the relative primality in Assumption 3, we get a decomposition of each cell D^{2n} into D^2's, corresponding to the eigenspaces for the T-action.

Note that the relative primality assumption also gives a restriction on the coefficient ring R. Assuming that α_i is not a zero-divisor in $H_T^*(pt;R)$ implies that R has torsion coprime to the orders of the groups $\pi_0(G)$, where G is a stabilizer group of a point in a cell. Indeed, if p is a prime dividing $|\pi_0(G)|$, then p must divide one of the weights α_i of that cell. Thus, we may apply the injectivity result to this T-space.

Altogether, these assumptions allow us to define a graph $\Gamma = (V,E)$ associated to X. The vertices of Γ are the isolated fixed points of the T-action on X. There is an edge connecting two vertices p and q if these fixed points lie in the closure of one of the D^2's described in Assumptions 3 and 4. Moreover, we associate to each edge the additional datum of a T-weight α_e, given by the weight of T acting on that D^2. We can interpret such a weight α_e as an element of $H_T^2(pt;R)$.

Definition 3.1 Given a graph $\Gamma = (V, E)$, with each edge $e \in E$ decorated by a T-weight α_e, we define the *graph cohomology*[7] of Γ to be

$$H^*(\Gamma) = \{f : V \to H_T^*(pt; R) \mid f(p) - f(q) \equiv 0 \pmod{\alpha_e} \text{ for every edge } e = (p, q)\}\,.$$

Note that when $\Gamma = (V, E)$ is the graph associated to X as described above, then this graph cohomology is a subring of the equivariant cohomology $H_T^*(X^T; R)$ of the fixed points X^T of X. We first prove a Lemma, which computes the T-equivariant cohomology of a 2-sphere. This is the starting point of the whole discussion.

Lemma 3.2 *Suppose T acts linearly and non-trivially on S^2 with weight α. If α is divisible by $p \in \mathbb{Z}$, assume that p is not a zero divisor in the coefficient ring R. Then the inclusion $(S^2)^T = \{N, S\} \hookrightarrow S^2$ induces injections $\imath^* : H_T^*(S^2, \{S\}) \to H_T^*(\{N\})$ and $\jmath^* : H_T^*(S^2) \to H_T^*(\{N, S\})$, with images*

$$\imath^*(H_T^*(S^2, \{S\})) = \left\{ g \in H_T^*(\{N\}) \;\middle|\; \alpha \mid g \right\}$$

and

$$\jmath^*(H_T^*(S^2)) = \left\{ (f, g) \in H_T^*(\{N\}) \oplus H_T^*(\{S\}) \;\middle|\; \alpha \mid f - g \right\}.$$

Proof: The first step is to prove the statement in relative cohomology. We first consider the case where $T = S^1$, acts on S^2 by $t \cdot z = t^a z$ for some $a \in \mathbb{Z}$, not a zero divisor in R. The cohomology $H_T^*(S^2, \{S\})$ is equal to $\tilde{H}^*(S^2 \times_{S^1} ES^1 / \{S\} \times_{S^1} ES^1)$, so we need to investigate the space $S^2 \times_{S^1} ES^1 / \{S\} \times_{S^1} ES^1$. Consider the map $S^2 \times_{S^1} ES^1 \to S^2/S^1 = [0, 1]$. Its fibers over the endpoints are BS^1 and over interior points are $B(\mathbb{Z}/a\mathbb{Z})$. Knowing this, we can write $S^2 \times_{S^1} ES^1 / \{S\} \times_{S^1} ES^1$ as $BS^1 \cup_f ([0, 1] \times B(\mathbb{Z}/a\mathbb{Z}))/(\{1\} \times B(\mathbb{Z}/a\mathbb{Z}))$ for some map $f : \{0\} \times B(\mathbb{Z}/a\mathbb{Z}) \to BS^1$. In other words, $S^2 \times_{S^1} ES^1 / \{S\} \times_{S^1} ES^1$ is the homotopy cofiber of f. Now consider the long exact sequence of the cofibration

$$\cdots \to \tilde{H}^*(Cof(f)) \xrightarrow{\imath^*} H^*(BS^1) \xrightarrow{f^*} H^*(B(\mathbb{Z}/a\mathbb{Z})) \to \tilde{H}^{*+1}(Cof(f)) \to \cdots\,.$$

We know $H^*(BS^1) = R[x]$ with $\deg(x) = 2$. Because a is not a zero-divisor in R, we also have $H^*(B(\mathbb{Z}/a\mathbb{Z}))) = R[y]/(ay)$ with $\deg(y) = 2$. Finally $f^*(x) = y$ up to a unit since otherwise it would contradict Claim 2.3 that $\tilde{H}^*(Cof(f))$ is evenly graded. Thus,

$$Im(\imath^*) = Ker(f^*) = \begin{cases} aR & \text{if } * = 2n \text{ and } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

which is precisely what we wanted to show.

Now we consider relative case for general T. In this case the torus T can always be decomposed as $T = T' \times S^1$, where T' acts trivially on S^2 and S^1 acts by $a \in \mathbb{Z}$, as in the previous case. Then by the Künneth theorem, we have the following diagram

[7] We apologize for the bad terminology; this is *not* a cohomology theory for graphs.

$$\begin{tikzcd}
H_T^*(S^2,\{S\}) \arrow[r, "\imath^*"] \arrow[d, equal] & H_T^*(\{N\}) \arrow[d, equal] \\
H_{T'}^*(pt)\otimes H_{S^1}^*(S^2,\{S\}) \arrow[r, "1\otimes\imath_{S^1}^*"] & H_{T'}^*(pt)\otimes H_{S^1}^*(\{N\}),
\end{tikzcd}$$

and so $Im(\imath^*) = H_{T'}^*(pt)\otimes Im(\imath_{S^1}^*)$, which is again what we want. Now we turn to the non-relative computation. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_T^*(S^2,\{S\}) \arrow[r] \arrow[d, "\imath^*"] & H_T^*(S^2) \arrow[r] \arrow[d, "\jmath^*"] & H_T^*(\{S\}) \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & H_T^*(\{N,S\},\{S\}) \arrow[r] & H_T^*(\{N,S\}) \arrow[r] & H_T^*(\{S\}) \arrow[r] & 0
\end{tikzcd}$$

Both the top and bottom sequences split, and therefore $Im(\jmath^*) = Im(\imath^*)\oplus H_T^*(pt)$, where $H_T^*(pt)\to H_T^*(\{N,S\})$ is the diagonal inclusion. It is now straight forward to check

$$\jmath^*(H_T^*(S^2)) = \left\{(f,g)\in H_T^*(\{N\})\oplus H_T^*(\{S\})\,\middle|\, \alpha\mid f-g\right\}.$$

□

We need a similar result for a $2n$-sphere. This is the technical heart of the proof of Theorem 3.4.

Lemma 3.3 *Suppose T acts linearly on S^{2n} with $n>1$, and suppose that the weights $\alpha_1,\ldots,\alpha_n$ are pairwise relatively prime over $H_T^*(pt;R)$. If α_i is divisible by $p\in\mathbb{Z}$, assume that p is not a zero divisor in R. Then the inclusion $(S^{2n})^T=\{N,S\}\hookrightarrow S^{2n}$ induces an injection $\jmath^*: H_T^*(S^{2n},\{S\})\to H_T^*(\{N\})$, with image*

$$\jmath^*(H_T^*(S^{2n},\{S\})) = \left\{g\in H_T^*(\{N\})\,\middle|\,\alpha_i\mid g\ \forall i\right\} \tag{3.1}$$

Proof: First, we check that the image of $\jmath^*$ is contained in the right hand side of (3.1). This is true because we can factor $\jmath^*$ in n different ways,

$$H_T^*(S^{2n},\{S\}) \longrightarrow H_T^*(S^2,\{S\}) \xrightarrow{\imath^*} H_T^*(\{N\}).$$

Thus, by Lemma 3.2, the image of $\jmath^*$ does land in the right hand side.

Now we show that $\jmath^*$ maps onto the right hand side of (3.1). We note that S^{2n} is the n-fold smash product $S^{2n}=\bigwedge_{i=1}^n S^2$ of 2-spheres. Therefore, it is possible to use the external cup product to multiply relative cohomology classes $y_i\in H_T^*(S^2,\{S\})$ to define a class in $H_T^*(S^{2n},\{S\})$. Choose a class $g\in H_T^*(\{N\})$ satisfying $\alpha_i\mid g$ for all i. By the relative primality assumption, we conclude that $\prod\alpha_i\mid g$, and so we can write

$$g=\beta\cdot\left(\prod\alpha_i\right).$$

Since the α_i are generator of the images $\imath^*(H_T^*(S^2,\{S\}))\subseteq H_T^*(\{N\})$, by Lemma 3.2, there exist classes $h_i\in H_T^*(S^2,\{S\})$ satisfying $\imath^*(h_i)=\alpha_i$. Therefore, the element $\beta\cdot h_1\smile\ldots\smile h_n\in$

$H^*(S^{2n}, \{S\})$ satisfies

$$\jmath^*(\beta \cdot h_1 \smile \ldots \smile h_n) = \beta \cdot \imath^*(h_1) \cdots \imath^*(h_n) = g.$$

Hence $\jmath^*$ is onto the image described in (3.1), completing the proof. $\square$

Theorem 3.4 *Let X be a T-space satisfying Assumptions 1 through 4. Then the map*

$$\iota^* : H_T^*(X; R) \to H_T^*(X^T; R)$$

is an injection, and its image is equal to the graph cohomology $H^*(\Gamma) \subseteq H_T^*(X^T; R)$, i.e.

$$H_T^*(X; R) \cong H^*(\Gamma). \tag{3.2}$$

Proof: Assumptions 1-4 imply that the hypotheses of Theorem 2.2 hold, and so we conclude that ι^* is an injection.

We now show that the image $\iota^*(H_T^*(X; R))$ is contained in $H^*(\Gamma, \alpha)$. Let κ be a class in $H_T^*(X; R)$, and let $\iota^*(\kappa)$ be its image in $H_T^*(X^T; R)$. We denote by $\iota_p^*(\kappa)$ the further restriction of κ to a single fixed point $p \in X^T$. To show that $\iota^*(\kappa)$ is in the graph cohomology, it suffices to check that for each edge $(p, q) \in E$, we have the relation

$$\iota_p^*(\kappa) - \iota_q^*(\kappa) \equiv 0 \pmod{\alpha_e}.$$

This follows by Lemma 3.2 from the fact that the restriction of κ to the S^2 joining p and q must be an equivariant class in $H_T^*(S^2; R)$.

We now introduce some notation. As before, we consider the filtration of the cell complex X by X_i, the set of the first i cells. This induces a filtration on the graph $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma$, where $\Gamma_i = (V_i, E_i)$ has vertices $V_i = (X_i)^T$ and edges $E_i = \{(p, q) \in E \mid p, q \in V_i\}$. We will now prove that the image of the equivariant cohomology $H_T^*(X)$ is in fact equal to $H^*(\Gamma)$. We will prove this by an inductive argument on the cells. For $X_1 = \{pt\}$, the result is immediate since both $H_T^*(X_1)$ and $H^*(\Gamma_1)$ are equal to $H_T^*(pt)$. Now assume the result is known for X_{i-1}. We wish to prove the result for X_i.

We claim that there is an exact sequence in graph cohomology

$$0 \longrightarrow H^*(\Gamma_i, \Gamma_{i-1}) \longrightarrow H^*(\Gamma_i) \xrightarrow{r_i} H^*(\Gamma_{i-1}) \longrightarrow 0 \tag{3.3}$$

where by $H^*(\Gamma_i, \Gamma_{i-1})$, we mean the *relative graph cohomology* defined by

$$H^*(\Gamma_i, \Gamma_{i-1}) := \{f \in H^*(\Gamma_i) \mid f(p) = 0 \text{ for all } p \in V_{i-1}\}.$$

Note that this is not a general property of graph cohomology, but only holds for those graphs coming from cell complexes. The relative graph cohomology consists of exactly those elements in $H^*(\Gamma_i)$ whose supports are concentrated on $V_i \backslash V_{i-1}$. The map $r_i : H^*(\Gamma_i) \to H^*(\Gamma_{i-1})$ is given by the restriction map $f \mapsto f|_{V_{i-1}}$. The kernel of r_i is $H^*(\Gamma_i, \Gamma_{i-1})$ by definition. Therefore, to show

the exactness of the sequence (3.3), it suffices to show that r_i is surjective. To do this, we will use the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & H^*(\Gamma_i, \Gamma_{i-1}) \arrow[r] & H^*(\Gamma_i) \arrow[r, "r_i"] & H^*(\Gamma_{i-1}) & \\ 0 \arrow[r] & H_T^*(X_i, X_{i-1}) \arrow[r] \arrow[u] & H_T^*(X_i) \arrow[r] \arrow[u] & H_T^*(X_{i-1}) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{3.4}$$

The bottom sequence comes from the long exact sequence of relative cohomology, which automatically splits into short exact sequences, as before. We know that the right vertical arrow is an isomorphism by induction. Since the bottom row is exact, a simple diagram chase implies that the restriction map r_i is surjective.

We will now show that the isomorphism holds at the level of X_i and Γ_i, i.e. that the middle vertical arrow is an isomorphism. By the Five Lemma, it suffices to show that the left vertical arrow is an isomorphism. This is the content of Lemma 3.3.

Finally, we note that

$$H_T^*(X) = \varprojlim H_T^*(X_i) = \varprojlim H^*(\Gamma_i) = H^*(\Gamma),$$

completing the proof. □

Remark 3.5 It is possible to recover the ordinary cohomology $H^*(X)$ from the T-equivariant cohomology by tensoring out the $H_T^*(pt)$. Namely,

$$H^*(X;R) = H_T^*(X;R) \otimes_{H_T^*(pt;R)} R.$$

Indeed, the Eilenberg-Moore spectral sequence $Tor_{H_T^*(pt)}(H_T^*(X), R) \Rightarrow H^*(X)$, coming from the pullback square

$$\begin{tikzcd} X \times ET \arrow[r] \arrow[d] & ET \arrow[d] \\ X \times_T ET \arrow[r] & BT \end{tikzcd}$$

collapses since $H_T^*(X)$ is a free $H_T^*(pt)$-module.

4 Module Generators of $H_T^*(X)$

We now make the further assumption that X is a CW complex, namely that the attaching maps glue $2n$ cells onto the $2(n-1)$ skeleton. In this section, we present canonical generators of $H_T^*(X)$ as a $H_T^*(pt, R)$-module. We will move freely between thinking of cohomology classes as either in $H_T^*(X)$ or in the graph cohomology $H^*(\Gamma)$.

The proofs of Theorems 2.2 and 3.4 hold verbatim if we use the filtration by skeleta X_p of the CW complex. Assume by induction that we have generators of $H_T^*(X_{p-1})$. To extend these to

$H_T^*(X_p)$, consider the short exact sequence

$$0 \longrightarrow H_T^*(X_p, X_{p-1}) \longrightarrow H^*(X_p) \longrightarrow H^*(X_{p-1}) \longrightarrow 0\ .$$

First, note that $H_T^*(X_p, X_{p-1}) \cong H_T^*(\bigvee S^p, *) \cong \bigoplus H_T^*(S^p, *)$ has a canonical choice of generators. Indeed, each $H_T^*(S^p, \{S\})$ has a canonical generator, namely the class whose restriction in $H_T^*(\{N\})$ is the product $\prod_i \alpha_i$ of the weights of the T-action on that sphere. The generators of $H^*(X_{p-1})$ have a unique lift to $H^*(X_p)$ because $H_T^k(X_p, X_{p-1})$ is zero for all $k < p-1$. These lifts, along with the images of the chosen generators of $H_T^*(X_p, X_{p-1})$, form a canonical set of generators of $H^*(X_p)$.

For each fixed point v, let C_v be the corresponding cell, and f_v be the corresponding generator of $H_T^*(X)$. Let $f_v(w)$ denote the restriction of f_v at the fixed point w. It is straightforward to check that the $\{f_v\}$ satisfy the following conditions.

1. Each f_v is homogeneous of degree $dim(C_v)$.
2. If $\dim(C_w) < \dim(C_v)$, then $f_v(w) = 0 \in H_T^*(pt)$.
3. If $\dim(C_w) = \dim(C_v)$, and $w \neq v$, then $f_v(w) = 0 \in H_T^*(pt)$.
4. $f_v(v) = \prod_{i=1}^{\dim(C_v)/2} \alpha_i \in H_T^*(pt)$, where the α_i are the labels of the edges connecting v to $\Gamma_{\dim(C_v)/2-1}$.

These conditions uniquely characterize the f_v. Indeed, let $\{f_v'\}$ be another set of generators satisfying the above conditions. Write them as $f_v' = \sum_w b_{vw} f_w$. By conditions 2 and 3, we have $b_{vw} = 0$ whenever $dim(w) \leq dim(v), w \neq v$. By condition 4, $b_{vv} = 1$. Now, if $dim(w) > dim(v)$, then $b_{vw} = 0$ because otherwise f_v would not be homogeneous.

Remark 4.1 In the situations where X is a manifold with a T-invariant Morse function f and the cell decomposition is constructed from the Morse flow with respect to f, then the above construction is the same as the following: given a fixed point v, consider the flow-up manifold Σ_v of codimension $dim(C_v)$. By Poincaré duality, it represents a cohomology class f_v satisfying exactly these conditions.

We illustrate these generators for some examples in the following section.

5 Grassmannians and flag varieties

We now turn our attention to the main examples that motivate the results in this paper. These are the based polynomial loop spaces ΩK of a compact simply connected semisimple Lie group K, which are sometimes called the affine Grassmannians. These fall into the more general category of examples of homogeneous spaces G/P for an arbitrary Kac-Moody group G (defined over $\mathbb{C}$) with P a parabolic subgroup. We will phrase the proofs in this section in a language that makes sense for this more general setting.

We will first consider in Section 5.1 the based loop spaces ΩK. As a homogeneous space, ΩK has an interpretation as a coadjoint orbit of LK. In this setting, the GKM graph can be embedded

in $\mathfrak{t}^*$ as the image of the 1-stratum under a T-moment map. The weights attached to the edges are encoded by their directions. Thus, this parallels the situation for the finite-dimensional coadjoint orbits. Throughout this section, we use the coefficient ring $R = \mathbb{Z}$.

5.1 Based loop spaces ΩK

We first quickly remind the reader of the definitions of the main characters in this section. The loop group LK of K is the set of polynomial loops

$$LK := \{\gamma : S^1 \to K\},$$

where the group structure is given by pointwise multiplication. By "polynomial," we mean that the loop is the restriction $S^1 = \{z \in \mathbb{C} : |z| = 1\} \to K$ of an algebraic map $\mathbb{C}^* \to K_{\mathbb{C}}$. The space of *based* polynomial loops is defined by

$$\Omega K = \{\sigma \in LK | \ \sigma(1) = 1\},$$

where, by abuse of notation, 1 is also the identity element in K. It is this space which is GKM space with respect to an appropriate torus action.

We first observe that LK acts transitively on ΩK as follows. For an element $\gamma \in LK, \sigma \in \Omega K$, we have

$$(\gamma \cdot \sigma)(z) = \gamma(z)\sigma(z)\gamma(1)^{-1}. \tag{5.1}$$

The last correction factor is required to insure that the new loop $\gamma \cdot \sigma$ is a *based* loop, i.e. that $(\gamma \cdot \sigma)(1) = 1 \in K$. This action is clearly transitive, and the stabilizer of the constant identity loop is K. Hence we may identify $\Omega K \cong LK/K$.

It is shown in [13, 8.3] that $\Omega K \cong LK/K$ is of the form G/P for the affine group $G = \widehat{LK_{\mathbb{C}}} \rtimes S^1$. Here, $LK_{\mathbb{C}}$ is the group of algebraic maps $\mathbb{C}^* \to K_{\mathbb{C}}$. The $\widehat{LK_{\mathbb{C}}}$ is the universal central extension of $LK_{\mathbb{C}}$, and the S^1 acts on $LK_{\mathbb{C}}$ by rotating the loop. The parabolic P is $\widehat{L^+K_{\mathbb{C}}} \rtimes S^1$, where $L^+K_{\mathbb{C}}$ is the subgroup of $LK_{\mathbb{C}}$ consisting of maps $\mathbb{C}^* \to K_{\mathbb{C}}$ that extend to maps $\mathbb{C} \to K_{\mathbb{C}}$. The identification is given by the action of LK on G/P by left multiplication. Then the stabilizer of the identity is $P \cap LK$. It is the set of polynomial maps $\mathbb{C}^* \to K_{\mathbb{C}}$ which extends over 0 and sends S^1 to K. A loop γ in $P \cap LK$ satisfies $\gamma(z) = \theta(\gamma(1/\bar{z}))$, where θ is the Cartan involution on $K_{\mathbb{C}}$. Therefore, since γ extends over zero, by setting $\gamma(\infty) = \theta(\gamma(0))$, it also extends over ∞. But then γ is an algebraic map from $\mathbb{P}^1$ to $K_{\mathbb{C}}$, and is therefore constant since $K_{\mathbb{C}}$ is affine. Hence $P \cap LK = K$.

The relevant torus action on ΩK is given by left multiplication by the maximal compact torus T_G in G. Note, however, that the center of G acts trivially. Thus we will restrict our attention to the action of the maximal torus $T_{K_{ad}}$ of $K_{ad} = K/Z(K)$ and the extra S^1 that rotates the loops. More explicitly, for $\gamma \in \Omega K, t \in T_K$, and $u \in S^1$,

$$(t, u) \cdot \gamma(z) = t\gamma(uz)\gamma(u)^{-1}t^{-1}.$$

5.2 Kac-Moody flag varieties

We now need to check that this space of based loops $\Omega K = G/P$ satisfies Assumptions 1-4 that are the hypotheses Theorem 3.4. In fact, the argument applies to any homogeneous space G/P

of a Kac-Moody group G, and P a parabolic, with the action of the maximal compact torus $T = T_G/Z(G)$ of $G/Z(G)$. It is shown in [2, 7, 9, 11] that G/P admits a CW decomposition

$$G/P = \coprod_{[w]\in W_G/W_P} B\tilde{w}P/P.$$

Here, W_G and W_P are respectively the Weyl groups of G and of (the semisimple part of) P, and $\tilde{w}$ is a representative of w in G. Each cell has a single T-fixed point $\bar{w} := \tilde{w}P/P$. These cells are T-invariant because T_G is a subgroup of B, and the center $Z(G)$ acts trivially. To understand the T-isotropy weights at each fixed point, we analyze the tangent space

$$T_{\bar{w}}B\bar{w} = T_{\bar{w}}B\tilde{w}P/P = \mathfrak{b}/\mathfrak{b}\cap \tilde{w}\mathfrak{p}\tilde{w}^{-1} = \mathfrak{b}/\mathfrak{b}\cap w\cdot \mathfrak{p}.$$

Therefore, the tangent space decomposes into 1-dimensional pieces, corresponding to the roots contained in $\mathfrak{b}$ but not in $w\cdot\mathfrak{p}$. In particular, the weights are all primitive and distinct. Now pick a root α in $\mathfrak{b}$ but not in $w\cdot\mathfrak{p}$. Let $e_\alpha, e_{-\alpha}$ be the standard root vectors for $\alpha, -\alpha$. Let $SL(2,\mathbb{C})_\alpha$ be the subgroup of G with Lie algebra spanned by $e_\alpha, e_{-\alpha}$, and $[e_\alpha, e_{-\alpha}]$ and let B_α be the Borel of $SL(2,\mathbb{C})_\alpha$ with Lie algebra spanned by e_α and $[e_\alpha, e_{-\alpha}]$. Let $\tilde{r}_\alpha := exp(\pi(e_\alpha - e_{-\alpha})/2)$ represent the element r_α of the Weyl group which is reflection along α. The α-eigenspace in the cell $B\bar{w}$ is $B_\alpha\bar{w} \cong \mathbb{C}$. Its closure is $SL(2,\mathbb{C})_\alpha\bar{w} \cong \mathbb{P}^1$, and the point at infinity is given by $\tilde{r}_\alpha wP/P = r_\alpha\cdot\bar{w} = \overline{r_\alpha w}$. This is another T-fixed point. Therefore G/P satisfies all the assumptions of Theorem 3.4 for $R = \mathbb{Z}$.

The GKM graph associated to G/P has vertices W_G/W_P, with an edge connecting $[w]$ and $[r_\alpha w]$ for all reflections r_α in W_G. The weight label on such an edge is α. It turns out that it is possible to embed this GKM graph in $\mathfrak{t}^*$, the dual of the Lie algebra of T, in such a way that the direction of each edge is given by its label. To produce this embedding, we pick a point in $\mathfrak{t}_G^*$ whose W_G-stabilizer is exactly W_P, take its W_G-orbit, and draw an edge connecting any two vertices related by a reflection in W_G. This graph sits in a fixed level of $\mathfrak{t}_G^*$ (this is only relevant when G is of affine type) and can therefore be thought of as sitting in $\mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of $T = T_G/Z(G)$. Since $R = \mathbb{Z}$ and since all weights are primitive, checking the GKM conditions on the weights amounts to checking that no two weights from a given vertex are collinear.

5.3 Moment maps for ΩK

So far, we have only considered spaces of polynomial loops in K. However, our results still apply to other spaces of loops, such as smooth loops, 1/2-Sobolev loops, etc. Indeed, the polynomial loops are dense in these other spaces of loops [13, 3.5.3], [11]. By Palais' theorem [12, Theorem 12], these dense inclusions are weak homotopy equivalences, and the same holds for the Borel constructions $X \times_T ET$. The statement of Palais' theorem is unfortunately only stated for open subsets of vector spaces, but can easily be seen to hold for arbitrary manifolds by a familiar Mayer-Vietoris argument.

For ΩK, the embedded GKM graph can be produced as the image of the 1-stratum under an appropriate T-moment map. We first describe the symplectic structure on ΩK. We write it as a pairing on $L\mathfrak{k}$. It defines an invariant closed 2-form on LK which descends to ΩK. Let $X, Y \in L\mathfrak{k}$.

We set

$$\omega(X,Y) := \int_{S^1} \langle X(t), Y'(t) \rangle \, dt, \tag{5.2}$$

where $\langle , \rangle$ denotes an invariant bilinear form on $\mathfrak{k}$. The moment map $\mu : \Omega K \to \mathfrak{t}^*$ for the T action is given as follows. Let X denote an element of $\mathfrak{t}$, and let $\gamma \in LK$. We think of γ here as an element of LK, but the formula descends to ΩK. The X component of the moment map is given by

$$\mu^X(\gamma) = \int_{S^1} \langle X, \gamma'(t)\gamma(t)^{-1} \rangle \, dt. \tag{5.3}$$

The S^1-moment map is given by the *energy function*,

$$\Phi(\gamma) = \frac{1}{2}\int_{S^1} \|\gamma(t)^{-1}\gamma'(t)\|^2 dt. \tag{5.4}$$

The fixed points in ΩK of the $T \times S^1$-action are exactly the homomorphisms $S^1 \to T \subset K$, and the image of ΩK under the $T \times S^1$-moment map is the convex hull of the images of the fixed points [1]. See Figure 2 for the case $K = SU(2)$.

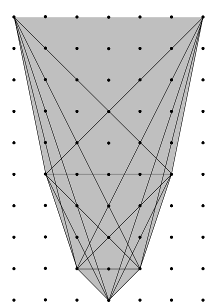

Figure 2: This is the moment polytope for the $T \times S^1$ action on $\Omega SU(2)$.

5.4 Loops in $SU(2)$

We now compute explicitly the ring structure of $H_T^*(\Omega SU(2); \mathbb{Z})$ using the moment map graph and the module generators f_v as constructed in Section 4. In this particular example, all the restrictions $f_v(w)$ at fixed points w happen to be elementary tensors in $H_T^*(\{w\}) \cong Sym(\Lambda)$, where $\Lambda = H_T^2(pt)$ is the weight lattice of T. This allows us to use the following convenient notation to represent the classes f_v. On every vertex w, we draw a bouquet of arrows $\beta_j \in \Lambda$ such that $f_v(w) = \prod \beta_j$. The vertices with no arrows coming out of them carry the class 0.

The first few module generators are illustrated in Figure 3. We call x the generator of degree 2, and express the others in terms of it. The arrows in the expressions denote elements in $H_T^2(pt) = \Lambda$.

The map $H_T^*(\Omega SU(2); \mathbb{Z}) \to H^*(\Omega SU(2); \mathbb{Z})$ is simply the map that sends the arrows to zero. And so, by tensoring out the $H_T^*(pt)$, we recover the well-known fact that the ordinary cohomology $H^*(\Omega SU(2); \mathbb{Z})$ is a divided powers algebra on a class in degree 2.

If instead, we take the coefficient ring $\mathbb{Q}$, then the cohomology of $\Omega SU(2)$ is isomorphic to that

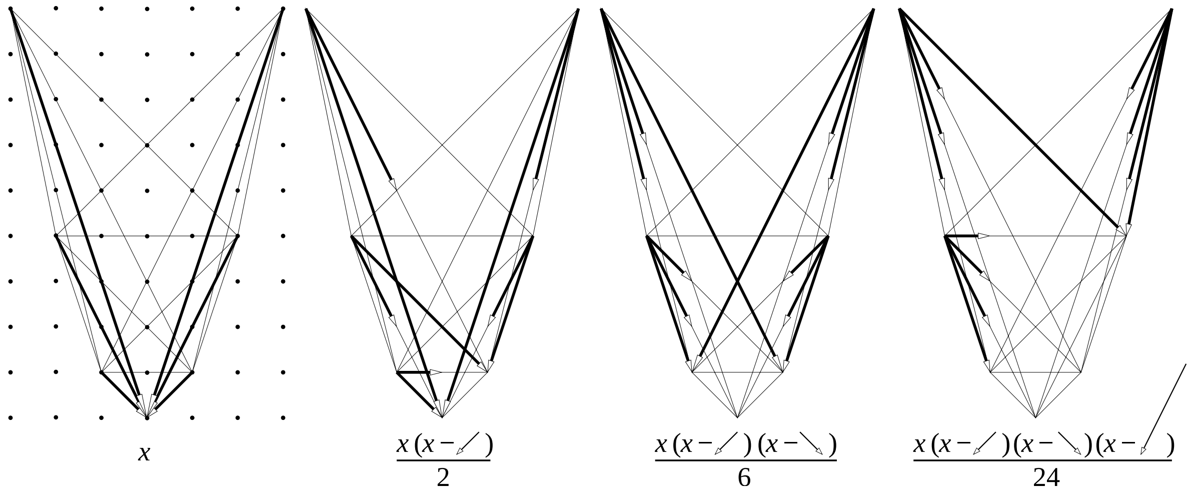

Figure 3: The degree 2,4,6, and 8 generators for $H_T^*(\Omega SU(2);\mathbb{Z})$. We draw in the lattice Λ in the leftmost figure.

of $\mathbb{C}P^\infty$. In fact, there is a T^2 action on $\mathbb{C}P^\infty$ that has the same moment map image as in Figure 2. This T-space satisfies Assumptions 1 through 4 over $\mathbb{Q}$, though not over $\mathbb{Z}$. Hence, for this action, $H_T^*(\mathbb{C}P^\infty;\mathbb{Q}) \cong H_T^*(\Omega SU(2);\mathbb{Q})$, but this is not true with $\mathbb{Z}$ coefficients.

5.5 A homogeneous space of type $A_1^{(4)}$

As another example of this type of computation, we let G be the affine group associated to the Cartan matrix

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

The group is $\widehat{LSL(3,\mathbb{C})}^{\mathbb{Z}/2\mathbb{Z}} \rtimes \mathbb{C}^*$, where the $\mathbb{Z}/2\mathbb{Z}$-action on $LSL(3,\mathbb{C})$ is given by precomposition with the antipodal map $z \mapsto -z$ on $\mathbb{C}^*$ and composition with the outer automorphism $A \mapsto (A^t)^{-1}$ of $SL(3,\mathbb{C})$.

We consider the homogeneous space G/P where the parabolic P has Lie algebra generated by $\mathfrak{b}$ and the negative of the simple short root. The degree 2, 4, 6, and 8 module generators in this case are illustrated in Figure 4. The denominator in the degree n-th module generator is given by $n!2^{\lfloor n/2 \rfloor}$.

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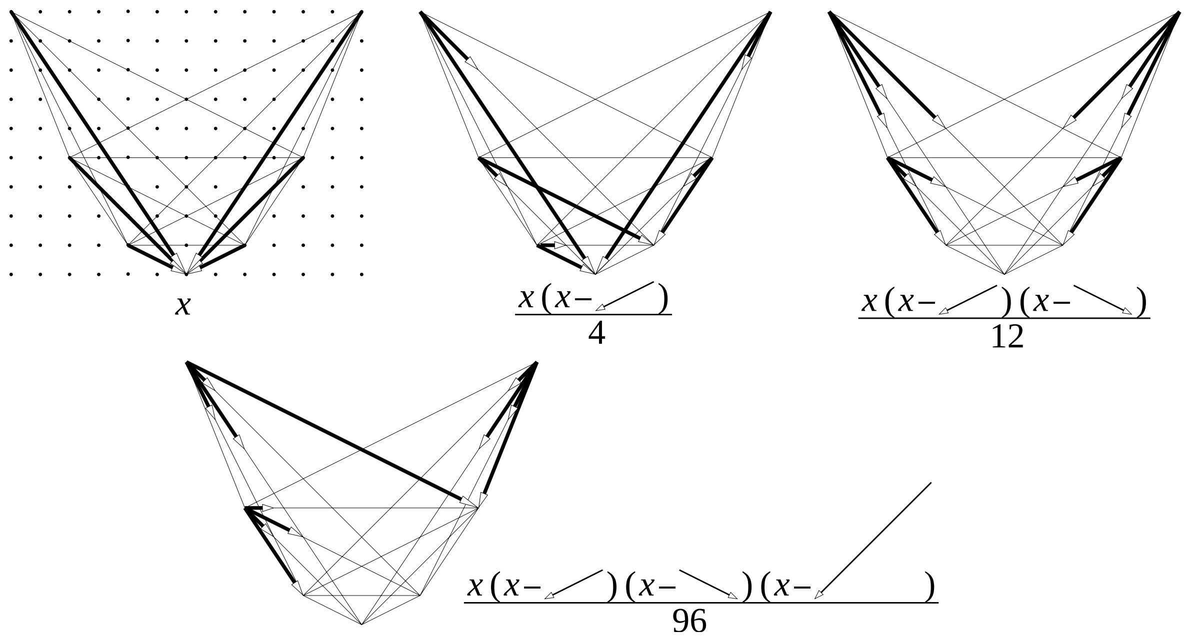

Figure 4: The degree 2, 4, 6, and 8 generators for $H_T^*(G/P;\mathbb{Z})$.

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